A simple Solution of Einstein's Field Equations

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Abstract

In this short paper we present a very simple but non-trivial vacuum solution of Einstein's field equations of gravitation.

Consider the metric

$$ds^2 = (dt^2 - dx^2 - dy^2 - dz^2) - (w\,dt + w\,dz)^2, \tag{1}$$

where $w = w(t, x, y, z)$ is a function of the Cartesian space-time coordinates x, y, z and t. Then a straightforward but longer computation[1] shows, that the components of the Riemann tensor R_{iklm} for this metric are given by

$$[R_{12lm}] = \frac{1}{2}\begin{bmatrix} 0 & X & Z & P \\ -X & 0 & 0 & -X \\ -Z & 0 & 0 & -Z \\ -P & X & Z & 0 \end{bmatrix} = [-R_{24lm}], \tag{2a}$$

$$[R_{13lm}] = \frac{1}{2}\begin{bmatrix} 0 & Z & Y & Q \\ -Z & 0 & 0 & -Z \\ -Y & 0 & 0 & -Y \\ -Q & Z & Y & 0 \end{bmatrix} = [-R_{34lm}], \tag{2b}$$

$$[R_{14lm}] = \frac{1}{2}\begin{bmatrix} 0 & P & Q & U \\ -P & 0 & 0 & -P \\ -Q & 0 & 0 & -Q \\ -U & P & Q & 0 \end{bmatrix}, \quad [R_{23lm}] = 0, \tag{2c}$$

where i, k, l, m range from 1 to 4 and

$$X = \frac{\partial^2 w^2}{\partial x^2}, \qquad Y = \frac{\partial^2 w^2}{\partial y^2}, \qquad Z = \frac{\partial^2 w^2}{\partial x \partial y}, \tag{3a}$$

$$U = \frac{\partial^2 w^2}{\partial t^2} + \frac{\partial^2 w^2}{\partial z^2} - 2\frac{\partial^2 w^2}{\partial t \partial z}, \tag{3b}$$

$$P = \frac{\partial^2 w^2}{\partial x \partial z} - \frac{\partial^2 w^2}{\partial x \partial t}, \tag{3c}$$

$$Q = \frac{\partial^2 w^2}{\partial y \partial z} - \frac{\partial^2 w^2}{\partial y \partial t}. \tag{3d}$$

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[1] In fact I have computed explicit formulas for the Riemann and Ricci tensors of a particular class of metrics containing the example presented here, but these results will be described in a further paper.

All remaining components of the Riemann tensor can be obtained from its well-known symmetries [3]. Consequently, the Ricci tensor R_{ik} for the metric is

$$[R_{ik}] = \frac{1}{2} \begin{bmatrix} (w^2-1)\,U - V & P & Q & w^2\,U - V \\ P & 0 & 0 & P \\ Q & 0 & 0 & Q \\ w^2\,U - V & P & Q & (w^2+1)\,U - V \end{bmatrix}, \tag{4}$$

where the abbreviation

$$V = X + Y = \frac{\partial^2 w^2}{\partial x^2} + \frac{\partial^2 w^2}{\partial y^2} \tag{5}$$

has been used. The metric therefore solves the vacuum field equations [1]

$$R_{ik} = 0, \tag{6}$$

if

$$U = V = P = Q = 0. \tag{7}$$

It is quite easy to find functions which satisfy (7). For example

$$\begin{aligned} w(t) &= \sqrt{a\,t + b}, \\ w(t,z) &= a\,(t+z) + b, \\ w(t,z) &= \sqrt{a\,(t^2 - z^2) + b}, \\ w(t,z) &= a\,\exp(b\,t + b\,z) + c, \\ w(t,x,y,z) &= \sqrt{a\,t + b\,x + c\,y + d\,z + e} \end{aligned}$$

do the job. However, in all these cases we also have

$$X = Y = Z = 0, \tag{8}$$

so that the Riemann tensors vanish and the metrics are merely parametrizations of the flat Minkowski space-time. But if we choose

$$w(x,y) = \sqrt{a\,\ln(b\,x^2 + b\,y^2) + c}, \tag{9}$$

which is a solution of the Laplace equation $V = 0$, then (7) is true, but (8) does not hold. We thus have found a simple non-trivial stationary and axially symmetric solution of the field equations.

Since the metric (1), (9) does probably not have any physical relevance, we do not go into a deeper discussion of the real parameters a, b, c here. It should merely be noticed, that the metric looks like an vacuum analogue of the Gödel solution [2], but this must be discussed in detail at another place.

References

[1] Albert Einstein, *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Physik 49, 769 – 822 (1916).

[2] Kurt Gödel, *An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation*, Reviews of Modern Physics 21, 447 – 450 (1949).

[3] Eberhard Klingbeil, *Tensorrechnung für Ingenieure*, Hochschultaschenbücher, Bibliographisches Institut, Mannheim (1966).