Representation of the Kerr Metric with a Complete Angular Momentum Vector

Ulric Kintzel*

March 31, 2022

Abstract

The Kerr metric describes the exterior region of a spinning black hole in terms of an angular momentum scalar. But from classical physics we know that the angular momentum is usually represented as a vector. Therefore, we investigate the question whether there is a modification of the Kerr metric in which a complete angular momentum vector occurs. We will find such a modified metric and examine its relationship to the original Kerr metric.

Keywords. Einstein's field equations, vacuum solution, Kerr metric.

1 Introduction

Shortly after the publication of Albert Einstein's general theory of relativity in 1916 [1], Karl Schwarzschild succeeded in finding a spherically symmetric solution of Einstein's field equations [4]. The Schwarzschild metric describes the exterior region of a massive physical object, a so-called black hole.

Afterwards further solutions of the field equations were found, such as the Gödel metric [2] or the Taub metric [5]. But none of these solutions brought new insights into the physics of black holes. It took until 1963 when Roy Kerr discovered a generalization of the Schwarzschild metric that describes the exterior region of a spinning black hole [3].

The Kerr metric contains an angular momentum parameter that represents the spin of the black hole as a scalar. But from classical physics we know that the angular momentum is usually represented as a vector. Therefore, we investigate the question whether there is a modification of the Kerr metric in which a complete angular momentum vector occurs.

The paper is organized as follows: We will work in Kerr-Schild coordinates and find such a modification of the Kerr metric. Afterwards we will examine the relationship between the original and the modified metric, and finally transform the found metric into spheroidal coordinates.

We assume that the reader is familiar with tensor calculus and use Einstein's summation convention without further reference. We also use a matrix notation that helps to formulate our findings in a condensed form.

*email: uk@ulric-kintzel.de

2 The Kerr metric

The Einstein field equations state that the 4-dimensional spacetime is a pseudo-Riemannian manifold in which

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}, \tag{1}$$

where $g_{\alpha\beta}$ is the metric tensor, $R_{\alpha\beta}$ is the Ricci tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the Ricci scalar, and $T_{\alpha\beta}$ is the energy-momentum tensor of the system under consideration. In vacuum the energy-momentum tensor is zero, so that the field equations simplify to

$$R_{\alpha\beta} = 0. \tag{2}$$

It is a hard mathematical problem to determine solutions of these nonlinear coupled differential equations for the unknown metric tensor $g_{\alpha\beta}$ [7]. The most important currently known vacuum solution is the Kerr metric which in Kerr-Schild coordinates has the form

$$G(t, x, y, z) = H - w(x, y, z) k(x, y, z) k(x, y, z)^T, \tag{3}$$

where the constant background metric is the Minkowski metric

$$H = H^{-1} = \text{diag}(+1, -1, -1, -1), \tag{4}$$

and the scalar function

$$w(x, y, z) = \frac{2mr^3}{r^4 + a^2 z^2} \tag{5}$$

and the vector

$$k(x, y, z) = \left(1, \frac{rx - ay}{r^2 + a^2}, \frac{ry + ax}{r^2 + a^2}, \frac{z}{r}\right)^T \tag{6}$$

contain functions of the spatial coordinates x, y, z. Here m is the mass of the central object, a is its scaled angular momentum, and r is a distorted radius which fulfills

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \tag{7}$$

and is explicitly given by

$$r(x, y, z) = \frac{\sqrt{2}}{2} \sqrt{x^2 + y^2 + z^2 - a^2 + \sqrt{\left(x^2 + y^2 + z^2 - a^2\right)^2 + 4a^2 z^2}}. \tag{8}$$

A brief introduction to the Kerr metric, which is stationary and does not depend on the time coordinate t, is given in [6].

Throughout the paper we use a matrix notation in which

$$G = [g_{\alpha\beta}], \quad H = [\eta_{\alpha\beta}] \qquad \text{and} \quad k = (k_\alpha) \tag{9}$$

contain the elements of the metric tensor. The background metric is used for raising (and lowering) indices

$$H^{-1} k = (\eta^{\alpha\beta} k_\beta) = (k^\alpha). \tag{10}$$

The Greek indices range from 1 to 4 and are assigned to the coordinates via $x^1 = t$, $x^2 = x$, $x^3 = y$, $x^4 = z$.

Since the angular momentum occurs as a scalar in the metric (3) and we know from classical physics that it is usually represented as a vector, the question naturally arises whether there is a modification of the metric in which a complete angular momentum vector $(c, b, a)^T$, corresponding to $(x, y, z)^T$, occurs? In fact, there is such a representation of the Kerr metric which we will show in the sequel. We proceed in two steps: First we derive the modified metric by an heuristic argument, and then we show that the found metric is a rotated version of the original metric (3).

3 The modified Kerr metric

We start our investigations by observing that the matrix

$$M = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & a & -b \\ 0 & -a & r & c \\ 0 & b & -c & r \end{bmatrix}^{-1} \tag{11}$$

$$= \frac{1}{r(r^2+a^2+b^2+c^2)} \begin{bmatrix} r^2+a^2+b^2+c^2 & 0 & 0 & 0 \\ 0 & r^2+c^2 & bc-ar & ac+br \\ 0 & bc+ar & r^2+b^2 & ab-cr \\ 0 & ac-br & ab+cr & r^2+a^2 \end{bmatrix},$$

transforms the vector

$$\ell = (r, x, y, z)^T \tag{12}$$

into a Kerr-like null vector

$$k(x,y,z) = M\ell = \begin{pmatrix} 1 \\ \dfrac{(r^2+c^2)x + (bc-ar)y + (ac+br)z}{r(r^2+a^2+b^2+c^2)} \\ \dfrac{(bc+ar)x + (r^2+b^2)y + (ab-cr)z}{r(r^2+a^2+b^2+c^2)} \\ \dfrac{(ac-br)x + (ab+cr)y + (r^2+a^2)z}{r(r^2+a^2+b^2+c^2)} \end{pmatrix}. \tag{13}$$

The null vector property

$$(H^{-1}k) \cdot k = 0, \tag{14}$$

furthermore implies

$$\frac{x^2+y^2+z^2}{r^2+a^2+b^2+c^2} + \frac{(cx+by+az)^2}{r^2(r^2+a^2+b^2+c^2)} = 1. \tag{15}$$

This equation generalizes condition (7) for the radius r that is now given by

$$r(x,y,z) = \frac{\sqrt{2}}{2}\sqrt{R^2 - A^2 + \sqrt{\left(R^2-A^2\right)^2 + 4D^2}}, \tag{16}$$

where

$$R^2 = x^2 + y^2 + z^2, \quad A^2 = a^2 + b^2 + c^2, \quad D = cx + by + az. \tag{17}$$

Finally, adapting the scalar function (5) accordingly, it turns out that

$$w(x, y, z) = \frac{2mr^3}{r^4 + (cx + by + az)^2} \tag{18}$$

is the correct choice to adjust the metric

$$G(t, x, y, z) = H - w(x, y, z)k(x, y, z)k(x, y, z)^T \tag{19}$$

(with (13), (18)) such that it solves the vacuum field equations (2).

This result covers the Kerr ($b = c = 0$) and the Schwarzschild metric ($a = b = c = 0$). But contrary to what we first thought, the new metric is actually just a rotated version of the original Kerr metric. In order to proof this, it suffices to consider the spatial part of the metrics (3) and (19), because the time coordinate t is not affected by the transformation.

4 The relationship between the metrics

We denote the position and angular momentum vector associated with the metric (19) as

$$\vec{x}_0 = (x, y, z)^T, \quad \vec{a}_0 = (c, b, a)^T, \tag{20}$$

and those associated with the metric (3) as

$$\vec{x}_1 = (\xi, \eta, \zeta)^T, \quad \vec{a}_1 = (0, 0, \alpha)^T. \tag{21}$$

Note that we use Greek letters for the vector components belonging to the original metric, and that

$$\alpha = |\vec{a}_0| = \sqrt{a^2 + b^2 + c^2} \tag{22}$$

is the Kerr angular momentum scalar. Now, let

$$u = \vec{a}_0 + \vec{a}_1, \quad \widehat{u} = \frac{u}{|u|}. \tag{23}$$

Then, by Rodrigues formula, the mapping

$$\operatorname{rot}(v, \varphi) = (\cos\varphi)\, v + (\sin\varphi)(\widehat{u} \times v) + (1 - \cos\varphi)(\widehat{u} \cdot v)\, \widehat{u} \tag{24}$$

rotates an arbitrary vector v about the axis $\widehat{u}$ by an angle φ. In particular, it holds that

$$\operatorname{rot}(\vec{a}_0, \pi) = \vec{a}_1, \tag{25}$$

and by rotating the corresponding position vector accordingly

$$\operatorname{rot}(\vec{x}_0, \pi) = \vec{x}_1, \tag{26}$$

we obtain the transformation rule

$$\vec{a}_1 = U\vec{a}_0, \quad \vec{x}_1 = U\vec{x}_0. \tag{27}$$

The involved transformation matrix (obtained from Eq. (24) by setting $\varphi = \pi$)

$$U = \frac{uu^T}{\alpha(a+\alpha)} - I, \quad I = \text{diag}(1,1,1) \tag{28}$$

is a symmetric orthogonal matrix

$$U^{-1} = U^T = U. \tag{29}$$

For the constants defined in Eq. (17) this means that

$$\begin{aligned} R^2 &= \vec{x}_0 \cdot \vec{x}_0 = \vec{x}_1 \cdot \vec{x}_1 = \xi^2 + \eta^2 + \zeta^2, \\ A^2 &= \vec{a}_0 \cdot \vec{a}_0 = \vec{a}_1 \cdot \vec{a}_1 = \alpha^2, \\ D &= \vec{a}_0 \cdot \vec{x}_0 = \vec{a}_1 \cdot \vec{x}_1 = \alpha\zeta, \end{aligned} \tag{30}$$

so that the transformation (27) leaves the Kerr radius (8), (16) unchanged. Moreover, we can rewrite the spatial part of the vector (13) as

$$k_0 = \frac{1}{r(r^2 + A^2)}(r^2\vec{x}_0 + D\vec{a}_0 + r\,\vec{a}_0 \times \vec{x}_0), \tag{31}$$

showing that

$$k_1 = Uk_0 = \frac{1}{r(r^2 + A^2)}(r^2\vec{x}_1 + D\vec{a}_1 + r\,\vec{a}_1 \times \vec{x}_1) \tag{32}$$

is just the spatial part of the vector (6). Finally, observing that the scalar functions (5), (18) both contain the invariant D^2, and that the orthogonal transformation U does not change the spatial part of the background metric H, we have shown that the metric (19) is a rotated version of the metric (3). We will therefore refer to the metric (19) as the "rotated" metric, and to the metric (3) as the "original" metric from now on.

This result is a bit of both: From the physical point of view it is disappointing that the more generally looking rotated metric does not provide new insights into the physics of spinning black holes. From the mathematical point of view it is satisfactory that the rotated metric is completely symmetric and that it shows all the components of the involved position and angular momentum vector.

In order to explain the meaning of the angular momentum vector more precisely, let us rewrite the 4-dimensional vector k from Eq. (13) as

$$k = e_1 + k_3, \quad k_3 = \frac{1}{r(r^2 + A^2)}(r^2\vec{x} + D\vec{a} + r\overrightarrow{ax}), \tag{33}$$

where k_3 is the 4-d extension of the vector k_0 from Eq. (31) (with added zero temporal component). The remaining vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} 0 \\ c \\ b \\ a \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}, \quad \overrightarrow{ax} = \begin{pmatrix} 0 \\ bz - ay \\ ax - cz \\ cy - bx \end{pmatrix}, \tag{34}$$

form a vector basis (provided that $\overrightarrow{ax} \neq 0$ and thus also $\vec{a} \neq 0$, $\vec{x} \neq 0$) in which $\overrightarrow{ax}$ contains the 3-d vector product of the vectors $\vec{a}$ and $\vec{x}$, so that the following scalar products apply

$$\begin{gathered} e_1 \cdot e_1 = 1, \quad e_1 \cdot \vec{a} = e_1 \cdot \vec{x} = e_1 \cdot \overrightarrow{ax} = 0, \\ \vec{a} \cdot \vec{a} = A^2, \quad \vec{x} \cdot \vec{x} = R^2, \quad \vec{a} \cdot \vec{x} = D, \\ \vec{a} \cdot \overrightarrow{ax} = \vec{x} \cdot \overrightarrow{ax} = 0, \quad \overrightarrow{ax} \cdot \overrightarrow{ax} = R^2 A^2 - D^2. \end{gathered} \tag{35}$$

The representation of the vector k in this basis reveals that the angular momentum vector $\vec{a}$ is directly involved in the Kerr metric. This basis is also a useful tool for performing calculations. For example, it helps to show that k_3 is a unit vector

$$k_3 \cdot k_3 = 1, \tag{36}$$

which can be verified directly using the scalar products (35), but follows more easily from Eq. (14) and

$$(H^{-1}k) \cdot k = (e_1 - k_3) \cdot (e_1 + k_3) = e_1 \cdot e_1 - k_3 \cdot k_3 = 0. \tag{37}$$

This property of the vector k_3 will play a role in the next discussion.

5 Transformation to spheroidal coordinates

At the beginning of our investigations we have discovered Eq. (13), which will now help to transform the rotated metric into spheroidal coordinates. For this purpose we leave the time coordinate t unchanged and obtain the spatial coordinates by demanding

$$\begin{pmatrix} 1 \\ k_x \\ k_y \\ k_z \end{pmatrix} = k = M\ell = k' = \begin{pmatrix} 1 \\ \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}, \tag{38}$$

which is possible due to Eq. (36) and gives

$$\begin{pmatrix} r \\ x \\ y \\ z \end{pmatrix} = \ell = M^{-1}k' = \ell' = \begin{pmatrix} r \\ (r\cos\varphi + a\sin\varphi)\sin\vartheta - b\cos\vartheta \\ (r\sin\varphi - a\cos\varphi)\sin\vartheta + c\cos\vartheta \\ (b\cos\varphi - c\sin\varphi)\sin\vartheta + r\cos\vartheta \end{pmatrix}. \tag{39}$$

The Jacobi matrix

$$T_{\rm sp} = \frac{\partial(t,x,y,z)}{\partial(t,r,\vartheta,\varphi)}$$

for the transformation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi\sin\vartheta & (r\cos\varphi + a\sin\varphi)\cos\vartheta + b\sin\vartheta & (a\cos\varphi - r\sin\varphi)\sin\vartheta \\ 0 & \sin\varphi\sin\vartheta & (r\sin\varphi - a\cos\varphi)\cos\vartheta - c\sin\vartheta & (r\cos\varphi + a\sin\varphi)\sin\vartheta \\ 0 & \cos\vartheta & (b\cos\varphi - c\sin\varphi)\cos\vartheta - r\sin\vartheta & -(b\sin\varphi + c\cos\varphi)\sin\vartheta \end{bmatrix},$$

which implies

$$H_{\rm sp} = T_{\rm sp}^T H T_{\rm sp} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & C & -E\sin\vartheta \\ 0 & C & -(r^2+A^2-E^2) & CE\sin\vartheta \\ 0 & -E\sin\vartheta & CE\sin\vartheta & -(r^2+a^2+B^2)\sin^2\vartheta \end{bmatrix},$$

as well as

$$k_{\rm sp}(r,\vartheta,\varphi) = T_{\rm sp}^T k = T_{\rm sp}^T k' = \begin{pmatrix} 1 \\ 1 \\ -C \\ E\sin\vartheta \end{pmatrix}, \quad w_{\rm sp}(r,\vartheta,\varphi) = \frac{2mr^3}{r^4+D^2} = \frac{2mr}{r^2+F^2}.$$

The constants A and D were already defined in Eq. (17) and the new abbreviations

$$\begin{aligned} B &= b\sin\varphi + c\cos\varphi, \quad C = c\sin\varphi - b\cos\varphi, \\ E &= a\sin\vartheta - B\cos\vartheta, \quad F = a\cos\vartheta + B\sin\vartheta \end{aligned} \tag{40}$$

are used. Moreover, the relationship between D and F follows from

$$D = \vec{a}\cdot\vec{x} = \vec{a}\cdot\ell = \vec{a}\cdot\ell' = rF. \tag{41}$$

In summary, the rotated metric in spheroidal coordinates is[1]

$$G_{\rm sp}(t,r,\vartheta,\varphi) = H_{\rm sp} - w_{\rm sp}(r,\vartheta,\varphi)k_{\rm sp}(r,\vartheta,\varphi)k_{\rm sp}(r,\vartheta,\varphi)^T \tag{42}$$

$$= \begin{bmatrix} 1-w & -w & wC & -wE\sin\vartheta \\ -w & -(1+w) & (1+w)C & -(1+w)E\sin\vartheta \\ wC & (1+w)C & -(U^2-E^2+(1+w)C^2) & (1+w)CE\sin\vartheta \\ -wE\sin\vartheta & -(1+w)E\sin\vartheta & (1+w)CE\sin\vartheta & -(U^2+wE^2)\sin^2\vartheta \end{bmatrix}.$$

Here we have used the fact that

$$b^2 + c^2 = B^2 + C^2 \tag{43}$$

to rewrite the (3, 3) element as

$$-G_{33} = r^2 + A^2 - E^2 + wC^2 = r^2 + a^2 + B^2 - E^2 + (1+w)C^2, \tag{44}$$

and we introduced the new abbreviation

$$U^2 = r^2 + a^2 + B^2. \tag{45}$$

In case of the original Kerr metric we have $b = c = 0$ which implies

$$B = C = 0, \quad E = a\sin\vartheta, \quad F = a\cos\vartheta, \quad U^2 = r^2 + a^2, \tag{46}$$

so that the transformed metric (42) takes on a massively simplified form in which six of the metric elements vanish ($C = 0$).

[1]Recall that we are using the "mostly minus" convention for the underlying Minkowski metric. Analogous results in the literature may have negated signs of the metric elements.

This is the basic form of spheroidal coordinates which with a further coordinate transformation could be mapped into the so-called Eddington-Finkelstein coordinates, and from these with a further transformation into the so-called Boyer-Lindquist coordinates. But of course there is no need to perform these transformation steps because we already know what the result is (when starting from the original Kerr metric). However, the discussion in this section shows how deeply involved the equation $M\ell = k$ is. Also, the transformation based on this equation leads to the "natural" abbreviations (40), (45) which seem to better represent the structure of the metric (42) than the standard abbreviations

$$\Sigma = r^2 + a^2 \cos^2 \vartheta, \quad \Delta = r^2 - 2mr + a^2$$

that frequently appear in the literature.

6 Conclusion

In this paper we have derived the metric (19) which describes the exterior region of a spinning black hole in terms of a complete angular momentum vector. We also showed how to transform this metric into spheroidal coordinates, giving the metric (42).

Since this metric is just a rotated version of the original Kerr metric (3) it does not directly provide new insights into the physics of spinning black holes. However, our investigations offer two possibilities for further research that may lead to new knowledge. These are:

(a) It would be interesting to understand the situations, in which the angular momentum vector depends on time, $\vec{a} = \vec{a}(t)$, or depends on space, $\vec{a} = \vec{a}(x, y, z)$, or does even depend on all coordinates, $\vec{a} = \vec{a}(t, x, y, z)$. What are the elements of the Ricci tensor $R_{\alpha\beta}$ in these cases, and what interpretation allows the associated energy-momentum tensor $T_{\alpha\beta}$?

(b) During the course of our research we first discovered the equation (13), $M\ell = k$, which then led to the metric (19). It would be interesting to investigate whether there are modifications of the matrix M which in the same way help to find new Ricci flat metrics. An interesting example for such a modification, inspired by an analogy to the relativistic angular momentum tensor, is

$$M = \begin{bmatrix} r & -A & -B & -C \\ A & r & a & -b \\ B & -a & r & c \\ C & b & -c & r \end{bmatrix}^{-1},$$

where A, B, C would be additional parameters of the possibly existing unknown metric.

Unfortunately, these questions cannot be answered easily, so that it will certainly take a while until the corresponding results are available.

Acknowledgements. I thank Professor Saul Teukolsky for reading a previous version of this manuscript, for asking whether the found metric is just a rotation of the original metric, and for his support in general.

References

[1] Albert Einstein, *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Physik 49, 769 – 822 (1916).

[2] Kurt Gödel, *An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation*, Reviews of Modern Physics 21, 447 – 450 (1949).

[3] Roy P. Kerr, *Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics*, Phys. Rev. Lett. 11, 237 – 238 (1963).

[4] Karl Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften, 189 – 196 (1916).

[5] Abraham H. Taub, *Empty space-times admitting a three parameter group of motions*, Annalen der Mathematik 53, 472 – 490 (1951).

[6] Matt Visser, *The Kerr spacetime: A brief introduction*, `https://arxiv.org/abs/0706.0622`, (2008).

For a comprehensive overview on solutions of the field equations, see:

[7] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, Eduard Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2nd edition (2003).