A Transformation Chart for the Kerr-Taub-NUT Metric and Instanton

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Abstract

In this paper we present the metric tensor of the Kerr-Taub-NUT metric and instanton in various coordinate systems and describe the relevant coordinate transformations in detail.

1 Introduction

In most publications the Schwarzschild metric [1], Taub-NUT metric [2, 3] and Kerr metric [4] are given in Boyer-Lindquist or Eddington-Finkelstein coordinates, and for the Kerr metric sometimes also Kerr-Schild coordinates are used. But there are hardly any presentations of these metrics in Cartesian coordinates, which may still be considered the "fundamental" coordinate system - of course not from the mathematical, but from the everyday experience point of view.

This is the starting point of this paper in which we will derive a complete transformation chart for the combined Kerr-Taub-NUT metric [5] which covers all the coordinate systems mentioned above. The notation is chosen such that it fits for metrics with Lorenzian $(-+++)$ and for instantons with Euclidean $(++++)$ signature.

2 Boyer-Lindquist Coordinates

The Lorenzian Kerr-Taub-NUT metric and the Euclidean Kerr-Taub-NUT instanton has in Boyer-Lindquist (spherical) coordinates the form

$$ds^2 = \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2\theta}{\Sigma} \left(a\ dt + P\ d\varphi \right)^2 + \varepsilon \frac{\Delta}{\Sigma} \left(dt + Q\ d\varphi \right)^2 , \tag{1}$$

where the metric functions are

$$\begin{aligned} \Delta &= r^2 - 2mr + \varepsilon \left(L^2 - a^2 \right), \\ \Sigma &= r^2 - \varepsilon \left(L + a\cos\theta \right)^2, \\ P &= \varepsilon r^2 - L^2 - a^2, \\ Q &= 2L\cos\theta - a\sin^2\theta, \end{aligned} \tag{2}$$

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and it holds that

$$\varepsilon\Sigma = P - a\,Q. \tag{3}$$

Here m, L, a denote the mass, magnetic monopole moment and angular momentum $J = ma$ of the field, and $\varepsilon = +1$ in the Euclidean or $\varepsilon = -1$ in the Lorenzian case, respectively.

The line element can also be written in the form $ds^2 = g_{ik}dx^i dx^k$ ($x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$) where the metric tensor $G_{\mathrm{BL}} = [g_{ik}]$ is

$$G_{\mathrm{BL}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\sin^2\theta}{\Sigma}\begin{pmatrix} a \\ 0 \\ 0 \\ P \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \\ P \end{pmatrix}^T + \varepsilon\frac{\Delta}{\Sigma}\begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}^T. \tag{4}$$

3 Cartesian Coordinates

Using the coordinate transformation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_{\mathrm{CT}} = \begin{pmatrix} t \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}_{\mathrm{BL}} \tag{5}$$

with

$$\frac{x^2+y^2}{r^2} + \frac{z^2}{r^2} = 1 \quad \text{or} \quad r^2 = x^2 + y^2 + z^2,$$

we obtain the metric tensor in Cartesian coordinates

$$\begin{aligned} G_{\mathrm{CT}} = {} & \frac{\Sigma}{r^2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T + \frac{\Sigma}{r^2 R^2}\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}^T \\ & + \frac{\Sigma}{r^2}\left(\frac{1}{\Delta} - \frac{1}{r^2}\right)\begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}\begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}^T + \frac{R^2}{\Sigma r^2}\begin{pmatrix} a \\ -y\,P/R^2 \\ x\,P/R^2 \\ 0 \end{pmatrix}\begin{pmatrix} a \\ -y\,P/R^2 \\ x\,P/R^2 \\ 0 \end{pmatrix}^T \\ & + \varepsilon\frac{\Delta}{\Sigma}\begin{pmatrix} 1 \\ -y\,Q/R^2 \\ x\,Q/R^2 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ -y\,Q/R^2 \\ x\,Q/R^2 \\ 0 \end{pmatrix}^T \end{aligned} \tag{6}$$

where

$$R^2 = r^2 - z^2 = x^2 + y^2, \quad \cos\theta = \frac{z}{r}, \quad \sin\theta = \frac{R}{r}.$$

The Jacobi matrix and its inverse

$$A_1 = \frac{\partial(t,x,y,z)_{\mathrm{CT}}}{\partial(t,r,\theta,\varphi)_{\mathrm{BL}}}\,, \qquad B_1 = A_1^{-1}$$

for the transformations to Boyer-Lindquist coordinates

$$A_1^T G_{\mathrm{CT}} A_1 = G_{\mathrm{BL}}, \qquad B_1^T G_{\mathrm{BL}} B_1 = G_{\mathrm{CT}}$$

are given by

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ 0 & \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x}{r} & \frac{xz}{R} & -y \\ 0 & \frac{y}{r} & \frac{yz}{R} & x \\ 0 & \frac{z}{r} & -R & 0 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & \frac{\cos\theta\cos\varphi}{r} & \frac{\cos\theta\sin\varphi}{r} & -\frac{\sin\theta}{r} \\ 0 & -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ 0 & \frac{xz}{r^2R} & \frac{yz}{r^2R} & -\frac{R}{r^2} \\ 0 & -\frac{y}{R^2} & \frac{x}{R^2} & 0 \end{bmatrix}.$$

4 Eddington-Finkelstein Coordinates

Using the coordinate transformation

$$\begin{pmatrix} t \\ r \\ \theta \\ \varphi \end{pmatrix}_{\mathrm{BL}} = \begin{pmatrix} t + \int P/\Delta\, dr \\ r \\ \theta \\ -\varphi - \int a/\Delta\, dr \end{pmatrix}_{\mathrm{EF}}, \tag{7}$$

we obtain the metric tensor in Eddington-Finkelstein coordinates

$$G_{\mathrm{EF}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\sin^2\theta}{\Sigma} \begin{pmatrix} a \\ 0 \\ 0 \\ -P \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 0 \\ -P \end{pmatrix}^T + \varepsilon\frac{\Delta}{\Sigma} \begin{pmatrix} 1 \\ \varepsilon\frac{\Sigma}{\Delta} \\ 0 \\ -Q \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon\frac{\Sigma}{\Delta} \\ 0 \\ -Q \end{pmatrix}^T \tag{8}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & (1+\varepsilon)\frac{\Sigma}{\Delta} & 0 & -Q \\ 0 & 0 & \Sigma & 0 \\ 0 & -Q & 0 & 0 \end{bmatrix} + \frac{\sin^2\theta}{\Sigma} \begin{pmatrix} a \\ 0 \\ 0 \\ -P \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 0 \\ -P \end{pmatrix}^T + \varepsilon\frac{\Delta}{\Sigma} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}^T.$$

The Jacobi matrix and its inverse

$$A_2 = \frac{\partial(t,r,\theta,\varphi)_{\mathrm{BL}}}{\partial(t,r,\theta,\varphi)_{\mathrm{EF}}}\,, \qquad B_2 = A_2^{-1}$$

for the transformations from Boyer-Lindquist coordinates

$$A_2^T G_{\mathrm{BL}} A_2 = G_{\mathrm{EF}}, \qquad B_2^T G_{\mathrm{EF}} B_2 = G_{\mathrm{BL}}$$

are given by

$$A_2 = \begin{bmatrix} 1 & P/\Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -a/\Delta & 0 & -1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 1 & -P/\Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -a/\Delta & 0 & -1 \end{bmatrix}.$$

5 Kerr-Schild Coordinates

Using the coordinate transformation

$$\begin{aligned} \begin{pmatrix} \tau \\ \xi \\ \eta \\ \zeta \end{pmatrix}_{\mathrm{KS}} &= \begin{pmatrix} t - r \\ \sqrt{r^2+a^2}\,\cos\left(\varphi - \arctan\frac{a}{r}\right)\sin\theta \\ \sqrt{r^2+a^2}\,\sin\left(\varphi - \arctan\frac{a}{r}\right)\sin\theta \\ r\cos\theta \end{pmatrix}_{\mathrm{EF}} \\ &= \begin{pmatrix} t - r \\ (r\cos\varphi + a\sin\varphi)\sin\theta \\ (r\sin\varphi - a\cos\varphi)\sin\theta \\ r\cos\theta \end{pmatrix}_{\mathrm{EF}} \end{aligned} \tag{9}$$

with

$$\frac{\xi^2+\eta^2}{r^2+a^2} + \frac{\zeta^2}{r^2} = 1 \quad \text{or} \quad r^2 = s^2 + \sqrt{s^4 + a^2\zeta^2} \quad \text{for} \quad s^2 = \frac{\xi^2+\eta^2+\zeta^2-a^2}{2}\,,$$

we obtain the metric tensor in Kerr-Schild coordinates

$$\begin{aligned} G_{\mathrm{KS}} = {} & \frac{\Sigma}{\rho^4}\left(r^2 + \frac{\omega^2 a^2 \zeta^2}{\Delta r^2}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T + \frac{\Sigma}{\rho^4}\left(\frac{r^2}{R^2} - \frac{a^2}{\Delta}\right) \begin{pmatrix} 0 \\ \xi \\ \eta \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \\ \eta \\ 0 \end{pmatrix}^T \\ & + \frac{\Sigma}{\rho^4}\left(\frac{\omega^2}{\Delta} - 1\right) \begin{pmatrix} 0 \\ \xi \\ \eta \\ \zeta \end{pmatrix} \begin{pmatrix} 0 \\ \xi \\ \eta \\ \zeta \end{pmatrix}^T + \frac{R^2}{\Sigma\rho^4 r^2} \begin{pmatrix} a\rho^2 \\ ar\xi + PV \\ ar\eta - PU \\ a(\omega^2+P)\frac{\zeta}{r} \end{pmatrix} \begin{pmatrix} a\rho^2 \\ ar\xi + PV \\ ar\eta - PU \\ a(\omega^2+P)\frac{\zeta}{r} \end{pmatrix}^T \\ & + \varepsilon\frac{\Delta}{\Sigma\rho^4} \begin{pmatrix} \rho^2 \\ br\xi + QV \\ br\eta - QU \\ (b\omega^2 + aQ)\frac{\zeta}{r} \end{pmatrix} \begin{pmatrix} \rho^2 \\ br\xi + QV \\ br\eta - QU \\ (b\omega^2 + aQ)\frac{\zeta}{r} \end{pmatrix}^T \end{aligned} \tag{10}$$

where

$$\omega^2 = r^2 + a^2, \quad R^2 = r^2 - \zeta^2 = \frac{r^2(\xi^2+\eta^2)}{\omega^2}, \quad \cos\theta = \frac{\zeta}{r}, \quad \sin\theta = \frac{R}{r},$$
$$U = \frac{r(r\xi - a\eta)}{\omega^2} + \frac{\xi\zeta^2}{R^2}, \quad V = \frac{r(r\eta + a\xi)}{\omega^2} + \frac{\eta\zeta^2}{R^2}, \quad b = 1 + \varepsilon\frac{\Sigma}{\Delta},$$
$$\rho^2 = r^2 + a^2\cos^2\theta = \frac{r^4 + a^2\zeta^2}{r^2}.$$

The Jacobi matrix and its inverse

$$A_3 = \frac{\partial(\tau,\xi,\eta,\zeta)_{\mathrm{KS}}}{\partial(t,r,\theta,\varphi)_{\mathrm{EF}}}, \qquad B_3 = A_3^{-1}$$

for the transformations to Eddington-Finkelstein coordinates

$$A_3^T G_{\mathrm{KS}} A_3 = G_{\mathrm{EF}}, \qquad B_3^T G_{\mathrm{EF}} B_3 = G_{\mathrm{KS}}$$

are given by

$$A_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \cos\varphi\sin\theta & (r\cos\varphi + a\sin\varphi)\cos\theta & -(r\sin\varphi - a\cos\varphi)\sin\theta \\ 0 & \sin\varphi\sin\theta & (r\sin\varphi - a\cos\varphi)\cos\theta & (r\cos\varphi + a\sin\varphi)\sin\theta \\ 0 & \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \dfrac{r\xi - a\eta}{\omega^2} & \dfrac{\xi\zeta}{R} & -\eta \\ 0 & \dfrac{r\eta + a\xi}{\omega^2} & \dfrac{\eta\zeta}{R} & \xi \\ 0 & \dfrac{\zeta}{r} & -R & 0 \end{bmatrix}$$

and

$$B_3 = \frac{1}{\rho^2}\begin{bmatrix} \rho^2 & r(r\cos\varphi + a\sin\varphi)\sin\theta & r(r\sin\varphi - a\cos\varphi)\sin\theta & \omega^2\cos\theta \\ 0 & r(r\cos\varphi + a\sin\varphi)\sin\theta & r(r\sin\varphi - a\cos\varphi)\sin\theta & \omega^2\cos\theta \\ 0 & (r\cos\varphi + a\sin\varphi)\cos\theta & (r\sin\varphi - a\cos\varphi)\cos\theta & -r\sin\theta \\ 0 & \dfrac{-r\cos\varphi + a\cos\varphi\cos^2\theta}{\sin\theta} & \dfrac{r\sin\varphi + a\sin\varphi\cos^2\theta}{\sin\theta} & -a\cos\theta \end{bmatrix}$$
$$= \frac{1}{\rho^2}\begin{bmatrix} \rho^2 & r\xi & r\eta & \omega^2\dfrac{\zeta}{r} \\ 0 & r\xi & r\eta & \omega^2\dfrac{\zeta}{r} \\ 0 & \dfrac{\xi\zeta}{R} & \dfrac{\eta\zeta}{R} & -R \\ 0 & -V & U & -a\dfrac{\zeta}{r} \end{bmatrix}.$$

6 Relationship between Cartesian and Kerr-Schild Coordinates

On the one hand we have (9)

$$\begin{pmatrix} \tau \\ \xi \\ \eta \\ \zeta \end{pmatrix}_{\text{KS}} = \begin{pmatrix} \tilde{t} - r \\ \sqrt{r^2 + a^2}\, \sin\theta \cos\left(\varphi - \arctan\frac{a}{r}\right) \\ \sqrt{r^2 + a^2}\, \sin\theta \sin\left(\varphi - \arctan\frac{a}{r}\right) \\ r\cos\theta \end{pmatrix}_{\text{EF}},$$

and on the other hand (5), (7) imply

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_{\text{CT}} = \begin{pmatrix} \tilde{t} + T(r) \\ r\sin\theta\cos\left(-\varphi - F(r)\right) \\ r\sin\theta\sin\left(-\varphi - F(r)\right) \\ r\cos\theta \end{pmatrix}_{\text{EF}},$$

where $\tilde{t}, r, \theta, \varphi$ are the Eddington-Finkelstein coordinates and

$$T(r) = \int \frac{P}{\Delta}\, dr\,, \qquad F(r) = \int \frac{a}{\Delta}\, dr\,.$$

From these equations it follows that

$$\begin{aligned} \tilde{t} &= \tau + r & &= t - T(r), \\ \theta &= \arcsin\sqrt{\frac{\xi^2+\eta^2}{r^2+a^2}} = \arccos\frac{\zeta}{r} & &= \arcsin\frac{\sqrt{x^2+y^2}}{r} = \arccos\frac{z}{r}\,, \\ \varphi &= \arctan\frac{\eta}{\xi} + \arctan\frac{a}{r} & &= -\arctan\frac{y}{x} - F(r), \end{aligned}$$

which leads to the mutual transformation equations

$$\begin{pmatrix} \tau \\ \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} t - T(r) - r \\ \hat{x}\, C(r) - \hat{y}\, S(r) \\ -\hat{y}\, C(r) - \hat{x}\, S(r) \\ z \end{pmatrix}, \qquad \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tau + T(r) + r \\ \hat{\xi}\, C(r) - \hat{\eta}\, S(r) \\ -\hat{\eta}\, C(r) - \hat{\xi}\, S(r) \\ \zeta \end{pmatrix}, \tag{11}$$

where

$$\hat{x} = x - \frac{a}{r}\, y, \quad \hat{y} = y + \frac{a}{r}\, x, \quad \hat{\xi} = \frac{r(r\xi - a\eta)}{r^2+a^2}\,, \quad \hat{\eta} = \frac{r(r\eta + a\xi)}{r^2+a^2}$$

and

$$C(r) = \cos F(r), \qquad S(r) = \sin F(r).$$

The Jacobi matrix and its inverse

$$A_4 = \frac{\partial(\tau,\xi,\eta,\zeta)}{\partial(t,x,y,z)}\,, \qquad B_4 = A_4^{-1}$$

for the transformations

$$A_4^T G_{\rm KS} A_4 = G_{\rm CT}, \qquad B_4^T G_{\rm CT} B_4 = G_{\rm KS}$$

can be found using the derivatives

$$T'(r) = \frac{P}{\Delta}\,, \quad F'(r) = \frac{a}{\Delta}\,, \quad C'(r) = -\frac{a}{\Delta}\,S(r), \quad S'(r) = \frac{a}{\Delta}\,C(r)$$

and the generalized chain rule for $r = \sqrt{x^2+y^2+z^2}$

$$f_x(x,y,z,r(x,y,z)) = f'(r)\;r_x(x,y,z) + f_x(x,y,z) = \frac{x}{r}\,f'(r) + f_x(x,y,z)$$

(analogously for y and z). It turns out that

$$A_4 = M \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{x}{r}\,Y & -\frac{a}{r} - \frac{y}{r}\,Y & -\frac{z}{r}\,Y \\ 0 & -\frac{a}{r} - \frac{x}{r}\,X & -1 - \frac{y}{r}\,X & -\frac{z}{r}\,X \\ 0 & 0 & 0 & 1 \end{bmatrix} N,$$

$$B_4 = \frac{r^2}{\rho^2}\,N^{-1} \begin{bmatrix} \frac{\rho^2}{r^2} & 0 & 0 & 0 \\ 0 & 1 + \frac{y}{r}\,X & -\frac{a}{r} - \frac{y}{r}\,Y & \frac{z}{r}\,Y - \frac{az}{r^2}\,X \\ 0 & -\frac{a}{r} - \frac{x}{r}\,X & -1 + \frac{x}{r}\,Y & -\frac{z}{r}\,X - \frac{az}{r^2}\,Y \\ 0 & 0 & 0 & \frac{\rho^2}{r^2} \end{bmatrix} M^{-1},$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C(r) & S(r) & 0 \\ 0 & -S(r) & C(r) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad M^{-1} = M^T,$$

$$N = \begin{bmatrix} 1 & -\beta\,\frac{x}{r} & -\beta\,\frac{y}{r} & -\beta\,\frac{z}{r} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad N^{-1} = \begin{bmatrix} 1 & \beta\,\frac{x}{r} & \beta\,\frac{y}{r} & \beta\,\frac{z}{r} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$X = \frac{a\hat{x}}{\Delta} - \frac{ax}{r^2}\,, \quad Y = \frac{a\hat{y}}{\Delta} - \frac{ay}{r^2}\,, \quad \rho^2 = \frac{r^4 + a^2\zeta^2}{r^2}\,, \quad \beta = \frac{P}{\Delta} + 1.$$

In the case $a = 0$ the equations (11) simplify to

$$\begin{pmatrix} \tau \\ \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} t - T(r) - r \\ x \\ -y \\ z \end{pmatrix}, \qquad \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tau + T(r) + r \\ \xi \\ -\eta \\ \zeta \end{pmatrix}, \tag{12}$$

and the Jacobi matrices become

$$A_4 = \begin{bmatrix} 1 & -\beta\,\dfrac{x}{r} & -\beta\,\dfrac{y}{r} & -\beta\,\dfrac{z}{r} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 1 & \beta\,\dfrac{x}{r} & -\beta\,\dfrac{y}{r} & \beta\,\dfrac{z}{r} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\beta = \varepsilon\Sigma/\Delta + 1$ according to (2).

7 Special Case: The Taub-NUT Metric

By setting $a = 0$ in (2) we find

$$\Delta = r^2 - 2mr + \varepsilon L^2, \quad \Sigma = r^2 - \varepsilon L^2 = \varepsilon P, \quad Q = 2L\cos\theta$$

and

$$\omega^2 = \rho^2 = r^2, \quad U = \frac{\xi\, r^2}{R^2}, \quad V = \frac{\eta\, r^2}{R^2}, \quad b = \beta = 1 + \varepsilon\frac{\Sigma}{\Delta}\,.$$

Now let

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \frac{Q}{R^2} = \frac{2Lz}{rR^2} = \frac{2L\zeta}{rR^2}\,.$$

Then the corresponding metric tensors for the Taub-NUT metric are

$$G_{\mathrm{CT}} = \frac{\Sigma}{r^2}\,J + \frac{\Sigma}{\Delta}\frac{r^2-\Delta}{r^4}\begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}\begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}^T + \varepsilon\frac{\Delta}{\Sigma}\begin{pmatrix} 1 \\ -yK \\ xK \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ -yK \\ xK \\ 0 \end{pmatrix}^T,$$

$$G_{\mathrm{BL}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ 0 & 0 & 0 & \Sigma\sin^2\theta \end{bmatrix} + \varepsilon\frac{\Delta}{\Sigma}\begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}^T,$$

$$G_{\mathrm{EF}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & (1+\varepsilon)\dfrac{\Sigma}{\Delta} & 0 & -Q \\ 0 & 0 & \Sigma & 0 \\ 0 & -Q & 0 & \Sigma\sin^2\theta \end{bmatrix} + \varepsilon\frac{\Delta}{\Sigma}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}^T,$$

$$G_{\mathrm{KS}} = \frac{\Sigma}{r^2}\,J + \frac{\Sigma}{\Delta}\frac{r^2-\Delta}{r^4}\begin{pmatrix} 0 \\ \xi \\ \eta \\ \zeta \end{pmatrix}\begin{pmatrix} 0 \\ \xi \\ \eta \\ \zeta \end{pmatrix}^T + \varepsilon\frac{\Delta}{\Sigma}\begin{pmatrix} 1 \\ b\,\dfrac{\xi}{r} + \eta K \\ b\,\dfrac{\eta}{r} - \xi K \\ b\,\dfrac{\zeta}{r} \end{pmatrix}\begin{pmatrix} 1 \\ b\,\dfrac{\xi}{r} + \eta K \\ b\,\dfrac{\eta}{r} - \xi K \\ b\,\dfrac{\zeta}{r} \end{pmatrix}^T.$$

Note that G_{KS} can either be obtained by simplifying (10) or by applying (12) to G_{CT}. Furthermore note that for the Schwarzschild metric we additionally have $L = 0$ which implies

$$\Delta = r^2 - 2mr, \quad \Sigma = r^2, \quad Q = K = 0,$$

and allows to simplify the tensors accordingly.

8 Special Case: The Kerr Metric

By setting $L = 0$ in (2) we find

$$\Delta = r^2 - 2mr - \varepsilon a^2, \quad \Sigma = r^2 - \varepsilon a^2 \cos^2\theta, \quad P = \varepsilon r^2 - a^2, \quad Q = -a\sin^2\theta,$$

which allows to express the "P terms" in the metric tensors (6), (4), (8) as

$$\frac{R^2}{\Sigma r^2}\begin{pmatrix} a \\ -y\,P/R^2 \\ x\,P/R^2 \\ 0 \end{pmatrix}\begin{pmatrix} a \\ -y\,P/R^2 \\ x\,P/R^2 \\ 0 \end{pmatrix}^T = \varepsilon \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

$$+\frac{\varepsilon P}{r^2R^2}\begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix}^T - \frac{P}{\Sigma}\begin{pmatrix} 1 \\ -y\,Q/R^2 \\ x\,Q/R^2 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ -y\,Q/R^2 \\ x\,Q/R^2 \\ 0 \end{pmatrix}^T,$$

$$\frac{\sin^2\theta}{\Sigma}\begin{pmatrix} a \\ 0 \\ 0 \\ P \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \\ P \end{pmatrix}^T = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon P\sin^2\theta \end{bmatrix} - \frac{P}{\Sigma}\begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}^T,$$

$$\frac{\sin^2\theta}{\Sigma}\begin{pmatrix} a \\ 0 \\ 0 \\ -P \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \\ -P \end{pmatrix}^T = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon P\sin^2\theta \end{bmatrix} - \frac{P}{\Sigma}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}^T.$$

For the Cartesian coordinates (6) it also holds that

$$\frac{\Sigma}{r^2R^2}\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}^T + \frac{\varepsilon P}{r^2R^2}\begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix}^T = \frac{\varepsilon P}{r^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\varepsilon Q^2}{R^4}\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}^T.$$

Now let

$$\Lambda = -\frac{P - \varepsilon\Delta}{\Sigma} = -\frac{2\varepsilon mr}{r^2 - \varepsilon a^2\cos^2\theta} = -\frac{2\varepsilon mr^3}{r^4 - \varepsilon a^2 z^2} = -\frac{2\varepsilon mr^3}{r^4 - \varepsilon a^2\zeta^2}$$

and

$$K = \frac{Q}{R^2} = -\frac{a}{r^2}.$$

Then the corresponding metric tensors for the Kerr metric are

$$G_{\mathrm{CT}} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon P/r^2 & 0 & 0 \\ 0 & 0 & \varepsilon P/r^2 & 0 \\ 0 & 0 & 0 & \Sigma/r^2 \end{bmatrix} + \varepsilon K^2 \begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}^T$$

$$+ \frac{\Sigma}{\Delta} \frac{r^2 - \Delta}{r^4} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}^T + \Lambda \begin{pmatrix} 1 \\ -yK \\ xK \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -yK \\ xK \\ 0 \end{pmatrix}^T,$$

$$G_{\mathrm{BL}} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \dfrac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ 0 & 0 & 0 & \varepsilon P \sin^2\theta \end{bmatrix} + \Lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix}^T,$$

$$G_{\mathrm{EF}} = \begin{bmatrix} \varepsilon & 1 & 0 & 0 \\ 1 & (1+\varepsilon)\dfrac{\Sigma}{\Delta} & 0 & -Q \\ 0 & 0 & \Sigma & 0 \\ 0 & -Q & 0 & \varepsilon P \sin^2\theta \end{bmatrix} + \Lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}^T.$$

A similar procedure could also be applied to the Kerr-Schild coordinates. But an easier way to find the wanted result is to rewrite the Eddington-Finkelstein coordinates as

$$G_{\mathrm{EF}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -Q \\ 0 & 0 & \rho^2 & 0 \\ 0 & -Q & 0 & \omega^2 \sin^2\theta \end{bmatrix} + \Lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -Q \end{pmatrix}^T$$

$$+ (1+\varepsilon) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & -a^2\cos^2\theta & 0 \\ 0 & 0 & 0 & -a^2\sin^2\theta \end{bmatrix},$$

and to transform each term with (9) into

$$G_{\mathrm{KS}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \Lambda \begin{pmatrix} 1 \\ \hat{\xi}/r \\ \hat{\eta}/r \\ \zeta/r \end{pmatrix} \begin{pmatrix} 1 \\ \hat{\xi}/r \\ \hat{\eta}/r \\ \zeta/r \end{pmatrix}^T$$

$$+ (1+\varepsilon) \left(b_0 b_0^T + \frac{\Sigma}{\Delta}\, b_1 b_1^T - \frac{a^2\zeta^2}{r^2}\, b_2 b_2^T - \frac{a^2 R^2}{r^2}\, b_3 b_3^T \right).$$

Here b_i $(0 \le i \le 3)$ are the column vectors of the matrix B_3^T,

$$b_0 = \frac{1}{\rho^2}\begin{pmatrix} \rho^2 \\ r\xi \\ r\eta \\ \omega^2 \frac{\zeta}{r} \end{pmatrix}, \quad b_1 = \frac{1}{\rho^2}\begin{pmatrix} 0 \\ r\xi \\ r\eta \\ \omega^2 \frac{\zeta}{r} \end{pmatrix}, \quad b_2 = \frac{1}{\rho^2}\begin{pmatrix} 0 \\ \frac{\xi\zeta}{R} \\ \frac{\eta\zeta}{R} \\ -R \end{pmatrix}, \quad b_3 = \frac{1}{\rho^2}\begin{pmatrix} 0 \\ -V \\ U \\ -a\frac{\zeta}{r} \end{pmatrix}.$$

For the Schwarzschild metric we additionally have $a = 0$ which implies

$$\Delta = r^2 - 2mr, \quad \Sigma = \varepsilon P = \rho^2 = \omega^2 = r^2, \quad Q = K = 0,$$
$$\hat{\xi} = \xi, \quad \hat{\eta} = \eta, \quad \varepsilon + \Lambda = \varepsilon \frac{\Delta}{\Sigma},$$

and allows to simplify the tensors accordingly. The expressions for G_{CT}, G_{BL}, G_{EF} obtained in this way are obviously equal to those derived in the previous section, and for G_{KS} it is easy to verify that in both cases

$$G_{\mathrm{KS}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + (1+\varepsilon)\frac{\Sigma}{\Delta}\begin{pmatrix} 0 \\ \xi/r \\ \eta/r \\ \zeta/r \end{pmatrix}\begin{pmatrix} 0 \\ \xi/r \\ \eta/r \\ \zeta/r \end{pmatrix}^T + \left(1 + \varepsilon\frac{\Delta}{\Sigma}\right)\begin{pmatrix} 1 \\ \xi/r \\ \eta/r \\ \zeta/r \end{pmatrix}\begin{pmatrix} 1 \\ \xi/r \\ \eta/r \\ \zeta/r \end{pmatrix}^T.$$

9 Conclusion

This paper provides formulas for the metric tensors of the Kerr-Taub-NUT metric and instanton in four different coordinate systems, and describes the relevant coordinate transformations in detail. The relationship between the coordinate systems is altogether summarized in the following transformation chart.

$$\begin{array}{ccccc} (4) & & & & (6) \\ G_{\mathrm{BL}} & \leftarrow & (5) & - & G_{\mathrm{CT}} \\ | & & & & \uparrow \\ (7) & & & & (11) \\ \downarrow & & & & | \\ G_{\mathrm{EF}} & \leftarrow & (9) & - & G_{\mathrm{KS}} \\ (8) & & & & (10) \end{array}$$

The equations (4) – (11) are valid for arbitrary metric functions Δ, Σ, P, Q satisfying (3), which for example holds for the Kerr-Taub-bolt metric and instanton [6], where

$$P = \varepsilon r^2 - a^2 - \frac{L^4}{L^2 - a^2} \quad \text{and} \quad Q = 2L\cos\theta - a\sin^2\theta - \frac{aL^2}{L^2 - a^2}.$$

We also specialized the general formulas to the important cases of the Schwarzschild, Taub-NUT and Kerr metric, and provided complete sets of metric tensors with Lorenzian and Euclidean signature in these cases as well.

Even though sometimes stated otherwise, these investigations clearly show, that the Kerr-Schild coordinates are not Cartesian, unless $a = 0$. For a true Cartesian coordinate system it always holds that $r^2 = x^2 + y^2 + z^2$. From the misinterpretation of the Kerr-Schild coordinates it also comes, that the Boyer-Lindquist coordinates are often considered as some sort of oblate spheroidal coordinate system. This is also not true. Boyer-Lindquist coordinates are just simple spherical coordinates, but they need to be compared with the right Cartesian coordinate system.

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